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Application of MacWilliams' Theorem for Complete Weight Enumerators on Galois Fields ($GF(q)$)

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Abstract. This paper implements the extended MacWilliams theorem for linear codes, revisiting the MacWilliams theorem for the Complete Weight Enumerator (CWE) of $GF(q)$ codes with q = 3, 4, 5, and 7. Linear codes are a mathematical concept that can be described through the distribution of weights in each codeword. CWE serves to provide a complete representation of the symbols in each codeword. This article discusses how the MacWilliams theorem can be efficiently used to calculate the CWE of codes and optimize the design of error-correcting codes. The results of this research include the calculation of CWE for $GF(3), GF(4), GF(5)$, and $GF(7)$. This study contributes theoretically to the understanding of linear code structures while opening up opportunities for the development of more efficient error-correcting code algorithms in modern communication systems.

Keywords: *MacWilliams theorem, Complete Weight Enumerator (CWE), Galois Field* ($GF(q)$)

Abstrak. Pada artikel ini diimplementasikan teorema MacWilliams yang diperluas untuk kode linier, disajikan kembali teorema MacWilliams untuk Complete Weight Enumerator (CWE) dari kode GF(q) untuk q = 3, 4, 5, dan 7. Kode linier adalah suatu konsep matematika yang dapat dijelaskan melalui distribusi bobot dari setiap kata dalam kode tersebut. CWE berfungsi untuk memberikan gambaran lengkap tentang simbol dalam setiap kata kode. Artikel ini, membahas bagaimana teorema MacWilliams dapat digunakan secara efisien untuk menghitung CWE kode dalam $GF(q)$ dan mengoptimalkan desain kode koreksi kesalahan. Hasil penelitian mencakup perhitungan CWE pada $GF(3), GF(4), GF(5)$, dan $GF(7)$. Penelitian ini memberikan kontribusi teoritis dalam memahami struktur kode linier sekaligus membuka peluang untuk pengembangan algoritma kode koreksi kesalahan yang lebih efisien dalam sistem komunikasi moderen.

Kata Kunci*: teorema MacWilliams, Complete Weight Enumerator (CWE), Galois Field* ($GF(q)$)

1. Introduction

Complete Weight Enumerator (CWE) is a concept that is often the focus of research in *coding* theory. CWE includes not only the weight enumerator but also the frequency of symbol occurrence in the codeword, this advantage makes CWE a more effective analytical tool for understanding linear code structures. The development in CWE research is marked by the work of Shudi Yang, Xiangli Kong, and Xueying Shi (2021)[4] who developed CWE for linear codes over finite fields, their research focuses on the construction of secret sharing schemes by utilizing exponential representations to show the relationship between weight enumerators and CWE of linear codesC_{D_c} with the defining setD_c. The contribution of this research not only deepens the theoretical understanding of the structure of linear codes, but also opens up practical applications in the field of information security, particularly in the design of secret sharing schemes.

Further assessment of CWE applications was conducted in a previous study that examined the application of CWE to *two-weight codes.* The study showed that CWE can be effectively combined in the development of authentication codes and secret sharing schemes with more efficient access structures. The advantage of CWE lies in its ability to provide more detailed structural information than *weight enumerators*, thus enabling more accurate fraud probability calculations on authentication codes and improved *soft decision decoding* performance [3].

Another contribution made by Xiangli Kong and Shudi Yang (2019) [2] introduced a new approach to compute CWE of linear codes with two or three weights. Their approach not only enriches the theoretical foundation of coding, but also has a direct impact on improving access structure and security in various cryptographic systems. A series of related studies [1] [5] [6] [7] [8] [9] have examined the use and development of CWE in various aspects of coding theory and modern cryptographic applications.

One interesting aspect of CWE is the MacWilliams identity, which involves the code and its dual code, especially in the context of weight enumerators [10]. This identity is crucial in understanding the relationship between the original code and the dual code, as well as in applications related to error detection and correction. In this study, we apply MacWilliams' theorem to compute CWE on $GF(q)$ ranging from $GF(3)$ to $GF(7)$ by integrating the character values (χ) of the studied codes. This research is expected to make significant theoretical contributions and practical benefits in the development of modern coding theory.

2. Theory Review

In this section, we will discuss the basic concepts used in this study. The main focus of this study is the discussion of $GF(q)$, characters (χ) , MacWilliams theorem and the application of MacWilliams theorem to CWE. An understanding of these concepts is essential to obtain the results in this study.

2.1 Galois Field $(\boldsymbol{GF(q)})$

Definition 2.1 A field is a set of elements on which the operations of addition, subtraction, multiplication, and division (except division by 0 which is undefined) can be performed. Addition and multiplication must satisfy the commutative, associative, and distributive properties for each element α, β, γ in the field [10] .

Finite fields have a finite number of elements, the number of elements is called the order of the field. This field is known as a Galois field, named after its inventor [10] . Field$(F) = GF(q) = GF(p^m)$, where p is a prime number. The elements of $GF(q)$ are denoted by $\omega_0 = 0, \omega_1, \ldots, \omega_{q-1}$ in a fixed order.

2.2 Character $GF(q)$

Each element β of $GF(q)$ can be written in the form [10] :

$$\beta = \beta_0 + \beta_1\alpha + \beta_2\alpha^2 + \ldots + \beta_{m-1}\alpha^{m-1}$$

Or equivalently as $m - tuple$:

$$\beta = (\beta_0, \beta_1, \ldots, \beta_{m-1})$$

Where α is a primitive element of $GF(q)$ and $0 \leq \beta \leq p - 1$

Suppose L is the square root p of the complex number$e^{2\pi i}$. It is the square root of p^{th} unit, which is $\xi^p = e^{2\pi i} = 1$, while $\xi^l \neq 1$ for $0 < l < p$.

Definition 2.2 For every$\beta = (\beta_0, \ldots, \beta_{m-1}) \in GF(q)$, define $\chi\beta$ as the complex-valued mapping defined on $GF(q)$ [10]:

$$\chi\beta(\gamma) = \xi^{\beta_0\gamma_0 + \beta_1\gamma_1 + \ldots + \beta_{m-1}\gamma_{m-1}} \text{ ....(1)}$$

For $\gamma = (\gamma_0, \ldots, \gamma_{m-1}) \in GF(q)$, χ_β is referred to as a character from $GF(q)$.

2.3 Linear Code

Definition 2.3 Suppose $\mathbb{F}_q^n$ denotes a vector space consisting of all consecutive sets$n$ of elements coming from a finite field$\mathbb{F}_q$. A code $C[n, k]$ over $\mathbb{F}_q$ is defined as a subset of $\mathbb{F}_q^n$ that has dimension k [10] .

Example 2.3 A code $C[4,2]$ over $\mathbb{F}_2$ is a subset of $\mathbb{F}_2^n$ with dimension 2
$\mathbb{F} = \{0,1\}$
$n = 4$
$\mathbb{F}_2^4 = \{(0000), (0001), (0010), \ldots, (1111)\}$
Then $C = \{(0000), (1011), (0101), (1110)\}$

2.4 Weight Enumerator

A weight enumerator is a polynomial used in code theory to represent the distribution of weights. The weight enumerator provides information about the number of codes with a certain weight in the code [10].

Definition 2.4 Ai denotes the number of codes with weight i in code C.
Define a polynomial :

$$\sum_{i=0}^{n} A_i\, x^{n-i} y^i \ldots\ldots\ldots (2)$$

Is the weight enumerator of the code C and is denoted by $W_C(x, y)$.

2.5 MacWilliams Theorem

Theorem 2.5 If C is a linear binary $[n, k]$ code with dual code $C^\perp$ then,

$$W_{C^\perp} = \frac{1}{|C|} W_C(x + y, x - y) \ldots\ldots\ldots (3)$$

where, $|C| = 2^k$ is the number of codes C [10].

The equation is equivalent to :

$$\sum_{k=0}^{n} A_k^{`} x^{n-k} y^k = \frac{1}{|C|} \sum_{i=0}^{n} A_i (x+y)^{n-i} (x-y)^i \ldots\ldots\ldots (4)$$

Or

$$\sum_{u \in C^{\perp}} x^{n-wt(u)} y^{wt(u)} = \frac{1}{|C|} \sum_{u \in C} (x+y)^{n-wt(u)} (x-y)^{wt(u)} \ldots\ldots\ldots (5)$$

2.4 MacWilliams Theorem for Complete Weight Enumerator

If C is a linear code $[n, k]$ over $GF(q)$ with complete weight enumerator W_c , the complete weight enumerator for the dual code $C^{\perp}$ is [10] :

$$W_{c^{\perp}}(z_0, \ldots, z_r, \ldots, z_{q-1}) = \frac{1}{|C|} W_c \begin{pmatrix} \sum_{i=0}^{q-1} \chi_1 (\omega_0 \omega_i) z_s, \ldots, \\ \sum_{i=0}^{q-1} \chi_1 (\omega_1 \omega_i) z_s, \ldots \end{pmatrix} \ldots\ldots\ldots (6)$$

3. Results and Discussion

In this section, we will present the results obtained from applying MacWilliams' theorem to CWE for$GF(q)$, from$GF(3)$ to $GF(7)$. Here are the results obtained:

3.1 Results for $GF(3)$

Character value of $GF(3)$

$$GF(3) = \{0,1,2\}$$
$$\xi = \omega \quad = e^{2\pi i/3}$$
$$\omega^3 \quad = 1$$

Character values obtained from $GF(3)$:

$$\begin{array}{lll} \chi_0(0) = 1, & \chi_1(0) = 1, & \chi_2(0) = 1, \\ \chi_0(1) = 1, & \chi_1(1) = \omega, & \chi_2(1) = \omega^2, \\ \chi_0(2) = 1, & \chi_1(2) = \omega^2, & \chi_2(2) = \omega. \end{array}$$

According to the application of MacWilliams theorem for CWE and with the results of (χ) characters from $GF(3)$, the CWE equation is obtained:

$$W_{c^{\perp}}(z_0, z_1, z_2) = \frac{1}{|C|} W_c \left(\sum_{s=0}^{2} \chi_1 (\omega_0 \omega_s) z_s, \sum_{s=0}^{2} \chi_1 (\omega_1 \omega_s) z_s, \sum_{s=0}^{2} \chi_1 (\omega_2 \omega_s) z_s \right)$$

where, $\omega = e^{2\pi i/3}$ and $W_{c^\perp}$ are obtained by applying linear transformation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}$$

So that results in $3 - tuple\ (z_0, z_1, z_2) =$
$(1,1,1), (1, \omega, \omega^2), (1, \omega^2, \omega)$.

From$GF(3)$ obtained $\omega_0 = 1, \omega_1 = 1, \omega_2 = 2$, after the calculation of the application of the CWE equation$GF(3)$ and the calculation value of the character(χ_1) , then the equation CWE dual from $GF(3)$ obtained the result:

$$W_{c^\perp}(z_0, z_1, z_2) = \frac{1}{|C|} W_c(z_0 + z_1 + z_2, z_0 + \omega z_1 + \omega^2 z_2, z_0 + \omega^2 z_1 + \omega z_2)$$

3.2 Results for $GF(4)$

Character value of $GF(4)$

$$\begin{aligned} &GF(4) = \{0,1,a,a+1\} \\ &\xi = \omega \quad = e^{2\pi i/2} \\ &\omega \qquad = e^{2\pi i} \\ &\qquad\ \ = \cos\pi + i\sin\pi \\ &\qquad\ \ = -1 \end{aligned}$$

Obtained character values from $GF(4)$:

$$\begin{array}{llll} \chi_0(0) = 1, & \chi_1(0) = 1, & \chi_a(0) = 1, & \chi_{a+1}(0) = 1 \\ \chi_0(1) = 1, & \chi_1(1) = -1, & \chi_a(1) = 1, & \chi_{a+1}(1) = -1, \\ \chi_0(a) = 1, & \chi_1(a) = 1, & \chi_a(a) = -1, & \chi_{a+1}(a) = -1 \\ \chi_0(a+1) = 1, & \chi_1(a+1) = -1, & \chi_a(a+1) = -1, & \chi_{a+1}(a+1) = 1. \end{array}$$

According to the application of MacWilliams theorem for CWE and with the results of (χ) characters from $GF(4)$, the CWE equation is obtained:

$$W_{c^\perp}(z_0, z_1, z_2, z_3) = \frac{1}{|C|} W_c \left(\begin{array}{c} \sum_{s=0}^{3} \chi_1(\omega_0 \omega_s) z_s, \sum_{s=0}^{3} \chi_1(\omega_1 \omega_s) z_s, \sum_{s=0}^{3} \chi_1(\omega_2 \omega_s) z_s, \\ \sum_{s=0}^{3} \chi_1(\omega_3 \omega_s) z_s \end{array} \right)$$

where,$\omega = e^{2\pi i}$ and $W_{c^\perp}$ are obtained by applying linear transformation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

This results in $4 - tuple(z_0, z_1, z_2, z_3)$:
$(1,1,1,1), (1,-1,1,-1), (1,1,-1,-1), (1,-1,-1,1)$

From $GF(4)$ obtained $\omega_0 = 0, \omega_1 = 1, \omega_2 = a, \omega_3 = a + 1$, after the calculation of the application of the CWE equation $GF(4)$ and the character calculation value (χ_1), then the equation CWE dual from $GF(4)$ obtained the result:

$$W_{c^\perp}(z_0, z_1, z_2, z_3) = \frac{1}{|C|} W_c \begin{pmatrix} z_0 + z_1 + z_2 + z_3, z_0 - z_1 + z_2 - z_3, z_0 + z_1 - z_2 + z_3, \\ z_0 - z_1 + z_2 + z_3 \end{pmatrix}$$

3.3 Results for $GF(5)$

Character value of $GF(5)$

$$\begin{aligned} GF(5) &= \{0,1,2,3,4\} \\ \xi = \omega &= e^{2\pi i/5} \\ \omega^5 &= 1 \end{aligned}$$

Obtained character values from $GF(5)$:

$$\begin{array}{lllll} \chi_0(0) = 1, & \chi_1(0) = 1, & \chi_2(0) = 1, & \chi_3(0) = 1, & \chi_4(0) = 1 \\ \chi_0(1) = 1, & \chi_1(1) = \omega, & \chi_2(1) = \omega^2, & \chi_3(1) = \omega^3, & \chi_4(1) = \omega^4, \\ \chi_0(2) = 1, & \chi_1(2) = \omega^2, & \chi_2(2) = \omega^4, & \chi_3(2) = \omega, & \chi_4(2) = \omega^3, \\ \chi_0(3) = 1, & \chi_1(3) = \omega^3, & \chi_2(3) = \omega, & \chi_3(3) = \omega^4 & \chi_4(3) = \omega^2, \\ \chi_0(4) = 1, & \chi_1(4) = \omega^4, & \chi_2(4) = \omega^3, & \chi_3(4) = \omega^2 & \chi_4(4) = \omega. \end{array}$$

According to the application of MacWilliams theorem for CWE and with the results of (χ) characters from $GF(5)$, the CWE equation is obtained:

$$W_{c^\perp}(z_0, z_1, z_2, z_3, z_4) = \frac{1}{|C|} W_c \begin{pmatrix} \sum_{s=0}^{4} \chi_1(\omega_0\omega_s) z_s, \sum_{s=0}^{4} \chi_1(\omega_1\omega_s) z_s, \\ \sum_{s=0}^{4} \chi_1(\omega_2\omega_s) z_s, \\ \sum_{s=0}^{4} \chi_1(\omega_3\omega_s) z_s, \sum_{s=0}^{4} \chi_1(\omega_4\omega_s) z_s \end{pmatrix}$$

where, $\omega^5 = 1$ and $W_{c^\perp}$ are obtained by applying linear transformation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

This result in $5 - tuple\ (z_0, z_1, z_2, z_3, z_4)$:
$(1,1,1,1,1), (1, \omega, \omega^2, \omega^3, \omega^4), (1, \omega^2, \omega^4, \omega, \omega^3),$

$$(1, \omega^3, \omega, \omega^4, \omega^2), (1, \omega^4, \omega^3, \omega^2, \omega)$$

From $GF(5)$ obtained $\omega_0 = 0, \omega_1 = 1, \omega_2 = 2, \omega_3 = 3, \omega_4 = 4$, after doing the calculation of the application of the CWE equation $GF(5)$ and the character calculation value (χ_1) , then the equation CWE dual from $GF(5)$ obtained the result:

$$W_{c^\perp}(z_0, z_1, z_2, z_3, z_4) = \frac{1}{|C|} W_c \begin{pmatrix} z_0 + z_1 + z_2 + z_3 + z_4, \\ z_0 + \omega z_1 + \omega^2 z_2 + \omega^3 z_3 + \omega^4 z_4, \\ z_0 + \omega^2 z_1 + \omega^4 z_2 + \omega z_3 + \omega^3 z_4, \\ z_0 + \omega^3 z_1 + \omega z_2 + \omega^4 z_3 + \omega^2 z_4, \\ z_0 + \omega^4 z_1 + \omega^3 z_2 + \omega^2 z_3 + \omega_4^z \end{pmatrix}$$

3.4 Results for $GF(7)$

Character value of $GF(7)$

$$GF(5) = \{0,1,2,3,4,5,6\}$$
$$\xi = \omega \quad = e^{2\pi i/7}$$
$$\omega^7 \quad = 1$$

Obtained character values from $GF(7)$:

$\chi_0(0) = 1,\quad \chi_1(0) = 1,\quad \chi_2(0) = 1,\quad \chi_3(0) = 1,$
$\chi_4(0) = 1,\quad \chi_5(0) = 1,\quad \chi_6(0) = 1,$
$\chi_0(1) = 1,\quad \chi_1(1) = \omega,\quad \chi_2(1) = \omega^2,\quad \chi_3(1) = \omega^3,$
$\chi_4(1) = \omega^4,\quad \chi_5(1) = \omega^5,\quad \chi_6(1) = \omega^6,$
$\chi_0(2) = 1,\quad \chi_1(2) = \omega^2,\quad \chi_2(2) = \omega^4,\quad \chi_3(2) = \omega^6,$
$\chi_4(2) = \omega,\quad \chi_5(2) = \omega^3,\quad \chi_6(2) = \omega^5,$
$\chi_0(3) = 1,\quad \chi_1(3) = \omega^3,\quad \chi_2(3) = \omega^6,\quad \chi_3(3) = \omega^2,$
$\chi_4(3) = \omega^5,\quad \chi_5(3) = \omega,\quad \chi_6(3) = \omega^4,$
$\chi_0(4) = 1,\quad \chi_1(4) = \omega^4,\quad \chi_2(4) = \omega,\quad \chi_3(4) = \omega^5,$
$\chi_4(4) = \omega^2,\quad \chi_5(4) = \omega^6,\quad \chi_6(4) = \omega^3,$
$\chi_0(5) = 1,\quad \chi_1(5) = \omega^5,\quad \chi_2(5) = \omega^3,\quad \chi_3(5) = \omega,$
$\chi_4(5) = \omega^6,\quad \chi_5(5) = \omega^4,\quad \chi_6(5) = \omega^2,$
$\chi_0(6) = 1,\quad \chi_1(6) = \omega^6,\quad \chi_2(6) = \omega^5,\quad \chi_3(6) = \omega^4,$
$\chi_4(6) = \omega^3,\quad \chi_5(6) = \omega^2,\quad \chi_6(6) = \omega,$

According to the application of MacWilliams theorem for CWE and with the results of (χ) characters from $GF(7)$, the CWE equation is obtained:

$$W_{c^\perp}(z_0,z_1,z_2,z_3,z_4,z_5,z_6)=\frac{1}{|C|}W_c\begin{pmatrix}\sum_{s=0}^{6}\chi_1(\omega_0\omega_s)z_s,\sum_{s=0}^{6}\chi_1(\omega_1\omega_s)z_s,\\ \sum_{s=0}^{6}\chi_1(\omega_2\omega_s)z_s,\\ \sum_{s=0}^{6}\chi_1(\omega_3\omega_s)z_s,\sum_{s=0}^{6}\chi_1(\omega_4\omega_s)z_s,\\ \sum_{s=0}^{6}\chi_1(\omega_5\omega_s)z_s,\\ \sum_{s=0}^{6}\chi_1(\omega_6\omega_s)z_s\end{pmatrix}$$

where, $\omega^7=1$ and $W_{c^\perp}$ are obtained by applying linear transformation

$$\begin{bmatrix}1&1&1&1&1&1&1\\1&\omega&\omega^2&\omega^3&\omega^4&\omega^5&\omega^6\\1&\omega^2&\omega^4&\omega^6&\omega&\omega^3&\omega^5\\1&\omega^3&\omega^6&\omega^2&\omega^5&\omega&\omega^4\\1&\omega^4&\omega&\omega^5&\omega^2&\omega^6&\omega^3\\1&\omega^5&\omega^3&\omega&\omega^6&\omega^4&\omega^2\\1&\omega^6&\omega^5&\omega^4&\omega^3&\omega^2&\omega\end{bmatrix}\begin{bmatrix}z_0\\z_1\\z_2\\z_3\\z_4\\z_5\\z_6\end{bmatrix}$$

This results in $7-tuple:(z_0,z_1,z_2,z_3,z_4,z_5,z_6)$:

$(1,\omega,\omega^2,\omega^3,\omega^4,\omega^5,\omega^6),(1,\omega^2,\omega^4,\omega^6,\omega,\omega^3,\omega^5),(1,\omega^3,\omega^6,\omega^2,\omega^5,\omega,\omega^4),$
$(1,\omega^4,\omega,\omega^5,\omega^2,\omega^6,\omega^3),(1,\omega^5,\omega^3,\omega,\omega^6,\omega^4,\omega^2),(1,\omega^6,\omega^5,\omega^4,\omega^3,\omega^2,\omega)$

From$GF(7)$, the value of $\omega_0=0,\omega_1=1,\omega_2=2,\omega_3=3,\omega_4=4,\omega_5=5,\omega_6=6$ is obtained, after calculating the application of the CWE equation$GF(5)$ and the value of the character calculation(χ_1), the dual CWE equation from$GF(7)$ is obtained:

$$\begin{aligned}W_{c^\perp}(z_0,z_1,z_2,z_3,z_4,z_5,z_6)=\frac{1}{|C|}W_c(&z_0+z_1+z_2+z_3+z_4+z_5+z_6,\omega z_1+\omega^2z_2\\&+\omega^3z_3+\omega^4z_4+\omega^5z_5+\omega^6z_6,\omega^2z_1+\omega^4z_2\\&+\omega^6z_3+\omega z_4+\omega^3z_5+\omega^5z_6,\omega^3z_1+\omega^6z_2\\&+\omega^2z_3+\omega^5z_4+\omega z_5+\omega^4z_6,\omega^4z_1+\omega^2z_2\\&+\omega^5z_3+\omega^2z_4+\omega^6z_5+\omega^3z_6,\omega^5z_1+\omega^3z_2\\&+\omega z_3+\omega^6z_4+\omega^4z_5+\omega^2z_6,\omega^6z_1+\omega^5z_2\\&+\omega^4z_3+\omega^3z_4+\omega^2z_5+\omega^1z_6)\end{aligned}$$

4. Conclusion

This research successfully applies MacWilliams' theorem to calculate the Complete Weight Enumerator (CWE) on Galois Field $GF(q)$ from $GF(3)$ to $GF(7)$. Through a systematic approach using the character χ of the CWE equation from MacWilliams' theorem, this research succeeds in finding the dual CWE equation of $GF(q)$. The results show that each $GF(q)$ generates a linear transformation matrix and produces a character value which is then applied in MacWilliams theorem to relate the original CWE code with its dual code.

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